On nonlinear TAR processes and threshold estimation

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Abstract

We consider the problem of threshold estimation for autoregressive time series with a "space switching" in the situation, when the regression is nonlinear and the innovations have a smooth, possibly non Gaussian, probability density. Assuming that the unknown threshold parameter is sampled from a continuous positive prior density, we find the asymptotic distribution of the Bayes estimator. As usually in the singular estimation problems, the sequence of Bayes estimators is asymptotically efficient, attaining the minimax risk lower bound.

Key words and phrases: Bayes estimator, compound Poisson process, likelihood inference, limit distribution, nonlinear threshold models, singular estimation.

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1 Introduction

The simplest threshold autoregressive (TAR) process is the time series, generated by the recursion

$$X_{j+1} = \rho_1 X_j \, \mathbb{1}_{\{X_j<\theta\}} + \rho_2 X_j \, \mathbb{1}_{\{X_j \geq \theta\}} + \varepsilon_{j+1}, \qquad j = 0, \ldots, n-1,$$

where $\varepsilon_j \sim \mathcal{N}\left(0, \sigma^2\right)$ are i.i.d. random variables and $\rho_1 \neq \rho_2$ and σ^2 are known constants. The unknown *threshold* parameter $\theta \in \Theta = (\alpha, \beta)$ is to be estimated from the data $X^n = (X_0, X_1, \ldots, X_n)$. This model and some of its generalizations has been extensively studied during the last decades (see e.g. [1]-[5],[9] and the references therein). Particularly, much attention focused on the properties of the least squares (LS) estimator

$$\theta_n^* = \mathrm{argmin}_{\theta\in\Theta} \sum_{j=0}^{n-1} \left[X_{j+1} - \rho_1 X_j \, \mathbb{1}_{\{X_j<\theta\}} - \rho_2 X_j \, \mathbb{1}_{\{X_j \geq \theta\}}\right]^2 .$$

Assuming that $|\rho_1| \vee |\rho_2| < 1$ and thus that (X_j) is geometric mixing with the unique invariant density $\varphi(x, \theta)$, Chan [1] proved consistency of θ_n^* with the rate n (rather than

$\sqrt{n}$ as in regular problems) and showed that the limit distribution is related to certain compound Poisson process (see (2) below). Note that if $\varepsilon_1 \sim \mathcal{N}\left(0,\sigma^2\right)$, the LS estimator coincides with the maximum likelihood (ML) estimator.

This work continues the study of the Bayes estimator for the TAR models, initiated in [3] and developed further in [2] and [4] (see also [7] for the continuous time counterpart). We consider the following more general nonlinear TAR(1) model

$$X_{j+1} = h\left(X_j\right)\, \mathbb{1}_{\{X_j<\theta\}} + g\left(X_j\right)\, \mathbb{1}_{\{X_j\geq\theta\}} + \varepsilon_{j+1}, \qquad j=0,\dots,n-1, \tag{1}$$

where $h(x)$ and $g(x)$ are known functions, (ε_j) are i.i.d. random variables with a known density function $f\left(x\right) > 0, x\in\mathbb{R}$ and the initial condition X_0 is independent of (ε_j) and has a probability density $f_0(x)$.

Throughout we shall assume that the following conditions are in force

(a1) The parameter $\theta\in(\alpha,\beta)\equiv\Theta, -\infty<\alpha<\beta<\infty$ is sampled from the continuous positive prior density $p\left(\theta\right), \theta\in\Theta$.

(a2) The functions h and g are continuous and satisfy

$$\inf_{v\in\Theta}\left|\delta\left(v\right)\right| > 0, \qquad \delta\left(v\right) := g\left(v\right) - h\left(v\right).$$

(a3) The random variables $(\varepsilon_j)_{j\geq 1}$ are i.i.d. with a known continuous bounded density function $f\left(x\right)>0, x\in\mathbb{R}$

(a4) The functions $h\left(x\right), g\left(x\right)$ and $f\left(x\right)$ are such that the time series, generated by (1), is geometric mixing with the unique positive bounded invariant density $\varphi\left(x,\theta\right)$, i.e. for any measurable function $|\psi(x)|\leq 1$

$$\mathbf{E}\left|\mathbf{E}\big(\psi(X_j)|\mathcal{F}_i\big) - \int_{\mathbb{R}}\psi(x)\varphi(x,\theta)dx\right| \leq Cr^{|j-i|}, \quad j>i$$

with positive constants C and $r<1$.

(a5) The function

$$J(z) := \int_{-\infty}^{\infty}\left|\ln\frac{f\left(y+z\right)}{f\left(y\right)}\right| f\left(y\right)\,\mathrm{d}y, \quad \min_{\theta\in\Theta}\delta(\theta)\leq z\leq\max_{\theta\in\Theta}\delta(\theta)$$

is bounded.

The likelihood function of the sample X^n is given by

$$L\left(\theta,X^n\right) = f_0\left(X_0\right)\prod_{j=0}^{n-1} f\Big(X_{j+1} - h\left(X_j\right)\, \mathbb{1}_{\{X_j<\theta\}} - g\left(X_j\right)\, \mathbb{1}_{\{X_j\geq\theta\}}\Big),$$

and the Bayes estimator $\tilde{\theta}_n$ with respect to the mean square risk is the conditional expectation

$$\tilde{\theta}_n = \mathbf{E}\left(\theta | X^n\right) = \frac{\int_\Theta \theta\, p\left(\theta\right)\ L\left(\theta, X^n\right)\ \mathrm{d}\theta}{\int_\Theta p\left(\theta\right)\ L\left(\theta, X^n\right)\ \mathrm{d}\theta}.$$

Since the likelihood $L(\theta, X^n)$ is piecewise constant in θ, the estimate can be computed efficiently (see [3]).

The asymptotic properties of $(\tilde{\theta}_n)$ are formulated in terms of the following compound Poisson process

$$Z\left(u\right) = \begin{cases} \exp\left(\sum_{l=1}^{N_+(u)} \ln \frac{f\left(\varepsilon_l^+ + \delta(\theta_0)\right)}{f\left(\varepsilon_l^+\right)}\right), & u \geq 0, \\ \exp\left(\sum_{l=1}^{N_-(-u)} \ln \frac{f\left(\varepsilon_l^- - \delta(\theta_0)\right)}{f\left(\varepsilon_l^-\right)}\right), & u < 0. \end{cases} \tag{2}$$

Here θ_0 is the true value of the parameter, $\epsilon_l^{\pm}$ are independent random variables with the density function $f\left(x\right)$, $N_+\left(\cdot\right)$, $N_-\left(\cdot\right)$ are independent Poisson processes with the same intensity $\lambda = \varphi\left(\theta_0, \theta_0\right)$ ($Z(u) := 1$ on the sets $\{N_{\pm}(u) = 0\}$).

Define the random variable

$$\tilde{u} = \frac{\int_R u\, Z\left(u\right)\ \mathrm{d}u}{\int_R Z\left(u\right)\ \mathrm{d}u}.$$

As shown in [2] (see [6] for the general theory), we have the following lower bound on the mean square risk of an arbitrary sequence of estimates $(\bar{\theta}_n)$:

$$\underline{\lim}_{\delta\to 0}\ \underline{\lim}_{n\to\infty}\ \sup_{|\theta-\theta_0|<\delta} n^2 \mathbf{E}_\theta \left(\bar{\theta}_n - \theta\right)^2 \geq \mathbf{E}_{\theta_0} \tilde{u}^2,$$

and the Bayes estimates $(\widetilde{\theta}_n)$ are *efficient*, attaining this lower bound asymptotically. Our main result is the following

Theorem 1.1. *Under the conditions* (**a1**)*-*(**a5**)*, the sequence of estimates* $(\tilde{\theta}_n)$ *is consistent, the convergence in distribution*

$$n\left(\tilde{\theta}_n - \theta_0\right) \Longrightarrow \tilde{u}$$

holds and the moments converge:

$$\lim_{n\to\infty} n^p \mathbf{E}_{\theta_0} \left|\tilde{\theta}_n - \theta_0\right|^p = \mathbf{E}_{\theta_0} \left|\tilde{u}\right|^p, \quad p > 0.$$

Remark 1.1. *The assumption* (**a4**) *is often easy to check, using the standard ergodic theory as e.g. in [8]. The assumption* (**a5**) *is satisfied for many common densities. For example, for the Gaussian innovations* $\varepsilon_j \sim N\left(0,\sigma^2\right)$,

$$J\left(z\right)\leq \frac{z^2}{2\sigma^2}+\frac{\left|z\right|}{\sigma}.$$

In this case, the limit compound Poisson process $Z\left(u\right)$ *has Gaussian jumps:*

$$\ln \frac{f\left(\varepsilon_1^{\pm}\pm\delta\left(\theta_0\right)\right)}{f\left(\varepsilon_1^{\pm}\right)}=-\frac{\delta\left(\theta_0\right)^2}{2\sigma^2}\mp\frac{\delta\left(\theta_0\right)}{\sigma^2}\,\varepsilon_1^{\pm}\sim \mathcal{N}\left(-\frac{\delta^2\left(\theta_0\right)}{2\sigma^2},\frac{\delta^2\left(\theta_0\right)}{\sigma^2}\right).$$

Similarly the assumption (**a5**) *is checked for the Laplace density* $f\left(y\right)=\left(2\sigma\right)^{-1}e^{-\frac{\left|y\right|}{\sigma}}$ *and the limit process has jumps of the form*

$$\ln \frac{f\left(\varepsilon_1^{\pm}\pm\delta\left(\theta_0\right)\right)}{f\left(\varepsilon_1^{\pm}\right)}=\frac{1}{\sigma^2}\Big(\left|\varepsilon_1^{\pm}\right|-\left|\varepsilon_1^{\pm}\pm\delta(\theta_0)\right|\Big).$$

2 The Proof

We shall verify the conditions of the Theorem 1.10.2 in [6], where the properties of the Bayes estimators, announced in Theorem 1.1, are derived from the convergence of the normalized likelihood ratios

$$Z_n\left(u\right)=\frac{L\left(\theta_0+u/n,X^n\right)}{L\left(\theta_0,X^n\right)},\qquad u\in\mathbb{U}_n=\left[n\left(\alpha-\theta_0\right),n\left(\beta-\theta_0\right)\right]$$

to the limit process $Z\left(u\right),u\in\mathbb{R}$ and the two inequalities (9) and (10), presented below. The change of variables $\theta=\theta_0+u/n$ gives

$$\tilde\theta_n=\frac{\int_{\mathbb{U}_n}\left(\theta_0+\frac{u}{n}\right)\,p\left(\theta_0+\frac{u}{n}\right)\frac{L\left(\theta_0+\frac{u}{n},X^n\right)}{L\left(\theta_0,X^n\right)}\mathrm{d}u}{\int_{\mathbb{U}_n}p\left(\theta_0+\frac{u}{n}\right)\frac{L\left(\theta_0+\frac{u}{n},X^n\right)}{L\left(\theta_0,X^n\right)}\mathrm{d}u}=\theta_0+\frac{1}{n}\frac{\int_{\mathbb{U}_n}u\left[p\left(\theta_0\right)+o\left(1\right)\right]Z_n\left(u\right)\mathrm{d}u}{\int_{\mathbb{U}_n}\left[p\left(\theta_0\right)+o\left(1\right)\right]Z_n\left(u\right)\mathrm{d}u}.$$

Then, informally, we have

$$\tilde u_n=n\left(\tilde\theta_n-\theta_0\right)=\frac{\int_{\mathbb{U}_n}u\;Z_n\left(u\right)\mathrm{d}u}{\int_{\mathbb{U}_n}Z_n\left(u\right)\mathrm{d}u}+o\left(1\right)\Longrightarrow\frac{\int_{\mathbb{R}}u\;Z\left(u\right)\mathrm{d}u}{\int_{\mathbb{R}}Z\left(u\right)\mathrm{d}u}\equiv\tilde u.$$

Theorem 1.10.2 in [6] validates this convergence along with the convergence of moments. Similar program has been realized in the preceding works [3], [2] and [4].

Remark 2.1. *To avoid inessential technicalities, we shall assume that (X_j) is stationary, i.e. $X_0 \sim \varphi(\cdot, \theta_0)$. Due to the mixing property (**a4**), all the results below can be derived without stationarity assumption, along the same lines with minor adjustments (see [4] for details).*

Remark 2.2. *Below, C, C', c, C_p, etc. denote constants, whose values are not important and may change from line to line. We shall denote by $\mathbf{P}_\theta$ and $\mathbf{E}_\theta$ the probability and the expectation, corresponding to the particular value of the unknown parameter $\theta \in \Theta$ and set $\mathcal{F}_j := \sigma\{\varepsilon_i, i \le j\}$. The standard $O(\cdot)$ and $o(\cdot)$ notations will be used and we set $\sum_{i=k}^m(\ldots) = 0$ and $\prod_{i=k}^m(\ldots) = 1$ for $k > m$.*

2.1 Convergence of f.d.f.

We shall prove the convergence of the finite dimensional distributions:

$$\big(\ln Z_n(u_1), \ldots, \ln Z_n(u_d)\big) \Longrightarrow \big(\ln Z(u_1), \ldots, \ln Z(u_d)\big), \quad u \in \mathbb{R}^d, \tag{3}$$

following [4]. We shall restrict the consideration to $0 = u_0 < u_1 < \ldots < u_d$, leaving out the similar complementary case. To this end, note that the declared limit process $\ln Z(u)$ has independent increments and

$$\mathbf{E}_{\theta_0} \exp\left(\sum_{j=1}^d \mathbf{i}\lambda_j \Big(\ln Z(u_j) - \ln Z(u_{j-1})\Big)\right) = \\ \exp\left(\sum_{j=1}^d (u_j - u_{j-1})\varphi(\theta_0, \theta_0)\Big(\Psi(\lambda_j) - 1\Big)\right) =: e^{H(\lambda)}, \quad \lambda \in \mathbb{R}^d,$$

where (recall that $\delta := g - h$)

$$\Psi(\lambda_j) := \mathbf{E}_{\theta_0} \exp\left(\mathbf{i}\lambda_j \ln \frac{f\big(\varepsilon_1 + \delta\,(\theta_0)\,\big)}{f\big(\varepsilon_1\big)}\right).$$

Since $\ln Z(0) = 0$ a.s., (3) follows from the convergence of characteristic functions of the increments

$$\lim_n \mathbf{E}_{\theta_0} \exp\left(\sum_{j=1}^d \mathbf{i}\lambda_j \Big(\ln Z_n(u_j) - \ln Z_n(u_{j-1})\Big)\right) = e^{H(\lambda)}, \quad \lambda \in \mathbb{R}^d.$$

Let $m(x, \theta) := h(x)\mathbb{1}_{\{x<\theta\}} + g(x)\mathbb{1}_{\{x\ge\theta\}}$ and note that

$$m(x, \theta_0 + u_{j-1}/n) - m(x, \theta_0 + u_j/n) = \delta(x)\mathbb{1}_{\{x \in \mathbb{D}_j^n\}},$$

where $\mathbb{D}_j^n := [\theta_0 + u_{j-1}/n, \theta_0 + u_j/n)$. Let $\mathbb{B}_{j-1}^n := [\theta_0, \theta_0 + u_{j-1}/n)$, then

$$
\begin{aligned}
\ln Z_n(u_j) - \ln Z_n(u_{j-1}) = \sum_{k=0}^{n-1} \ln \frac{f\Big(X_{k+1} - m(X_k, \theta_0 + u_j/n)\Big)}{f\Big(X_{k+1} - m(X_k, \theta_0 + u_{j-1}/n)\Big)} = \\
\sum_{k=0}^{n-1} \ln \frac{f\Big(\varepsilon_{k+1} + m(X_k, \theta_0) - m(X_k, \theta_0 + u_j/n)\Big)}{f\Big(\varepsilon_{k+1} + m(X_k, \theta_0) - m(X_k, \theta_0 + u_{j-1}/n)\Big)} = \\
\sum_{k=0}^{n-1} \ln \frac{f\Big(\varepsilon_{k+1} + \delta(X_k)\mathbb{1}_{\{X_k \in \mathbb{B}_{j-1}^n\}} + \delta(X_k)\mathbb{1}_{\{X_k \in \mathbb{D}_j^n\}})\Big)}{f\Big(\varepsilon_{k+1} + \delta(X_k)\mathbb{1}_{\{X_k \in \mathbb{B}_{j-1}^n\}}\Big)} = \\
\sum_{k=0}^{n-1} \mathbb{1}_{\{X_k \in \mathbb{D}_j^n\}} \ln \frac{f\Big(\varepsilon_{k+1} + \delta(X_k)\mathbb{1}_{\{X_k \in \mathbb{B}_{j-1}^n\}} + \delta(X_k)\Big)}{f\Big(\varepsilon_{k+1} + \delta(X_k)\mathbb{1}_{\{X_k \in \mathbb{B}_{j-1}^n\}})\Big)} \stackrel{\dagger}{=} \\
\sum_{k=0}^{n-1} \mathbb{1}_{\{X_k \in \mathbb{D}_j^n\}} \ln \frac{f\big(\varepsilon_{k+1} + \delta(X_k)\big)}{f\big(\varepsilon_{k+1}\big)} =: \sum_{k=0}^{n-1} s_k^j
\end{aligned}
\tag{4}
$$

where the equality $\dagger$ holds $\mathbf{P}_{\theta_0}$-a.s., since $\mathbf{P}_{\theta_0}\Big(X_{k-1} \in \mathbb{B}_{j-1}^n \cap \mathbb{D}_j^n\Big) = 0$. Further, define

$$
S_n := \sum_{j=1}^d \lambda_j \Big(\ln Z_n(u_j) - \ln Z_n(u_{j-1})\Big) = \sum_{j=1}^d \lambda_j \sum_{k=0}^{n-1} s_k^j.
$$

We shall partition n terms of this sum into $n^{1/2}$ consecutive blocks of size $n^{1/2}$ and discard from each block its $n^{1/4}$ first entries. As we shall see, this does not alter the asymptotic distribution of S_n, but makes the blocks almost independent. Since in each block, the single event $\{X_k \in D_j^n\}$ occurs with probability of order $n^{1/2}$, the Poisson behavior emerges. To implement these heuristics, define

$$
S_{m,n} := \sum_{j=1}^d \lambda_j \sum_{k=(m-1)n^{1/2}+n^{1/4}}^{mn^{1/2}} s_k^j, \quad m = 1, ..., n^{1/2},
$$

and set $\widetilde{S}_n := \sum_{m=1}^{n^{1/2}} S_{m,n}$ (this is the sum, in which the $n^{1/4}$ entries of each block have been discarded). By the triangle inequality

$$
\begin{aligned}
\Big|\mathbf{E}_{\theta_0} e^{\mathbf{i}S_n} - e^{H(\lambda)}\Big| \le \Big|\mathbf{E}_{\theta_0} e^{\mathbf{i}S_n} - \mathbf{E}_{\theta_0} e^{\mathbf{i}\widetilde{S}_n}\Big| + \\
\Big|\mathbf{E}_{\theta_0} e^{\mathbf{i}\widetilde{S}_n} - \Big(\mathbf{E}_{\theta_0} e^{\mathbf{i}S_{1,n}}\Big)^{n^{1/2}}\Big| + \Big|\Big(\mathbf{E}_{\theta_0} e^{\mathbf{i}S_{1,n}}\Big)^{n^{1/2}} - e^{H(\lambda)}\Big|.
\end{aligned}
\tag{5}
$$

We shall show that all the terms on the right hand side vanish as $n \to \infty$. By stationarity and the assumption (**a5**),

$$
\begin{aligned}
&\Big|\mathbf{E}_{\theta_0} e^{\mathbf{i}S_n} - \mathbf{E}_{\theta_0} e^{\mathbf{i}\widetilde{S}_n}\Big| \le \mathbf{E}_{\theta_0}\Big|e^{\mathbf{i}S_n} - e^{\mathbf{i}\widetilde{S}_n}\Big| \le \mathbf{E}_{\theta_0}\Big|S_n - \widetilde{S}_n\Big| \le \\
&n^{3/4}\max_j|\lambda_j|\mathbf{E}_{\theta_0}\mathbb{1}_{\{X_0\in\mathbb{D}^n_j\}}\left|\ln\frac{f\big(\varepsilon_1+\delta(X_0)\big)}{f\big(\varepsilon_1\big)}\right| = \\
&n^{3/4}\max_j|\lambda_j|\mathbf{E}_{\theta_0}\mathbb{1}_{\{X_0\in\mathbb{D}^n_j\}}J\big(\delta(X_0)\big) \le \\
&n^{3/4}\max_j|\lambda_j|\frac{u_j-u_{j-1}}{n}\sup_{x\in\mathbb{R}}\varphi(x,\theta_0)\sup_{\theta\in\Theta}J(\delta(\theta)) \xrightarrow{n\to\infty} 0,
\end{aligned}
$$

i.e. the first term in (5) converges to zero.

Further, note that by the Markov property of (X_j) and (**a4**)

$$
\Big|\mathbf{E}_{\theta_0}\Big(e^{\mathbf{i}S_{\ell,n}}\big|\mathcal{F}_{(\ell-1)n^{1/2}}\Big) - \mathbf{E}_{\theta_0}e^{\mathbf{i}S_{1,n}}\Big| \le Cr^{n^{1/4}}, \quad \ell = 1,...,n^{1/2}
$$

and hence

$$
\begin{aligned}
&\left|\mathbf{E}_{\theta_0}e^{\mathbf{i}\widetilde{S}_n} - \Big(\mathbf{E}_{\theta_0}e^{\mathbf{i}S_{1,n}}\Big)^{n^{1/2}}\right| = \left|\mathbf{E}_{\theta_0}\prod_{m=1}^{n^{1/2}}e^{\mathbf{i}S_{m,n}} - \Big(\mathbf{E}_{\theta_0}e^{\mathbf{i}S_{1,n}}\Big)^{n^{1/2}}\right| = \\
&\left|\sum_{\ell=1}^{n^{1/2}}\left(\mathbf{E}_{\theta_0}\prod_{m=1}^{\ell}e^{\mathbf{i}S_{m,n}}\Big(\mathbf{E}_{\theta_0}e^{\mathbf{i}S_{1,n}}\Big)^{n^{1/2}-\ell} - \mathbf{E}_{\theta_0}\prod_{m=1}^{\ell-1}e^{\mathbf{i}S_{m,n}}\Big(\mathbf{E}_{\theta_0}e^{\mathbf{i}S_{1,n}}\Big)^{n^{1/2}-\ell+1}\right)\right| = \\
&\left|\sum_{\ell=1}^{n^{1/2}}\left(\mathbf{E}_{\theta_0}\prod_{m=1}^{\ell-1}e^{\mathbf{i}S_{m,n}}\Big(e^{\mathbf{i}S_{\ell,n}} - \mathbf{E}_{\theta_0}e^{\mathbf{i}S_{1,n}}\Big)\Big(\mathbf{E}_{\theta_0}e^{\mathbf{i}S_{1,n}}\Big)^{n^{1/2}-\ell}\right)\right| = \\
&\left|\sum_{\ell=1}^{n^{1/2}}\left(\mathbf{E}_{\theta_0}\prod_{m=1}^{\ell-1}e^{\mathbf{i}S_{m,n}}\Big(\mathbf{E}_{\theta_0}\Big(e^{\mathbf{i}S_{\ell,n}}\big|\mathcal{F}_{(\ell-1)n^{1/2}}\Big) - \mathbf{E}_{\theta_0}e^{\mathbf{i}S_{1,n}}\Big)\Big(\mathbf{E}_{\theta_0}e^{\mathbf{i}S_{1,n}}\Big)^{n^{1/2}-\ell}\right)\right| \le \\
&\sum_{\ell=1}^{n^{1/2}}\mathbf{E}_{\theta_0}\Big|\mathbf{E}_{\theta_0}\Big(e^{\mathbf{i}S_{\ell,n}}\big|\mathcal{F}_{(\ell-1)n^{1/2}}\Big) - \mathbf{E}_{\theta_0}e^{\mathbf{i}S_{1,n}}\Big| \le Cn^{1/2}r^{n^{1/4}} \xrightarrow{n\to\infty} 0.
\end{aligned}
$$

It is left to show that the last term in (5) converges to zero. Let $\mathbb{D}^n = \bigcup_{j=1}^d \mathbb{D}^n_j$ and introduce the following events

$$
\begin{aligned}
&A_0 := \bigcap_{\ell\le n^{1/2}}\{X_\ell \notin \mathbb{D}^n\}, \quad A_1 := \bigcup_{j=1}^{d}\bigcup_{\ell=0}^{n^{1/2}}A_{\ell,j}, \qquad A_{2+} := \Big(A_0\cup A_1\Big)^c \\
&A_{k,j} := \{X_k \in \mathbb{D}^n_j\} \cap \bigcap_{\ell\le n^{1/2},\ell\ne k}\{X_\ell \notin \mathbb{D}^n\}.
\end{aligned}
$$

In words, A_0 is the event, on which none of the first $n^{1/2}$ samples falls in any of $\mathbb{D}_j^n$'s, A_1 is the event of having exactly single sample visiting one of $\mathbb{D}_j^n$'s, etc. On the event $A_{k,j}$,

$$S_{1,n} = \sum_{i=1}^d \lambda_i \sum_{\ell=n^{1/4}}^{n^{1/2}} \mathbb{I}_{\{X_\ell \in \mathbb{D}_i^n\}} \ln \frac{f\big(\varepsilon_{\ell+1} + \delta(X_\ell)\big)}{f\big(\varepsilon_{\ell+1}\big)} = \lambda_j \ln \frac{f\big(\varepsilon_{k+1} + \delta(X_k)\big)}{f\big(\varepsilon_{k+1}\big)}$$

and, since $\{X_k \in \mathbb{D}_j^n\} = A_{k,j} \biguplus \Big(\{X_k \in \mathbb{D}_j^n\} \cap \bigcup_{\ell\neq k}\{X_\ell \in \mathbb{D}^n\}\Big)$,

$$\mathbf{E}_{\theta_0} e^{\mathbf{i}S_{1,n}} \mathbb{I}_{A_1} = \sum_{j=1}^d \sum_{k=0}^{n^{1/2}} \mathbf{E}_{\theta_0} e^{\mathbf{i}S_{1,n}} \mathbb{I}_{A_{k,j}} = \sum_{j=1}^d \sum_{k=0}^{n^{1/4}-1} \mathbf{P}_{\theta_0}(A_{k,j}) + \tag{6}$$
$$\sum_{j=1}^d \sum_{k=n^{1/4}}^{n^{1/2}} \mathbf{E}_{\theta_0} \exp\left(\mathbf{i}\lambda_j \ln \frac{f\big(\varepsilon_{k+1} + \delta(X_k)\big)}{f\big(\varepsilon_{k+1}\big)}\right)\Big(\mathbb{I}_{\{X_k \in \mathbb{D}_j^n\}} - \mathbb{I}_{\{X_k \in \mathbb{D}_j^n\} \cap \bigcup_{\ell\neq k}\{X_\ell \in \mathbb{D}^n\}}\Big).$$

By continuity of $\varphi(x, \theta_0)$ and $\delta(x)$,

$$\mathbf{P}_{\theta_0}(A_{k,j}) \le \mathbf{P}_{\theta_0}(X_k \in \mathbb{D}_j^n) = \frac{u_j - u_{j-1}}{n}\varphi(\theta_0, \theta_0) + o(n^{-1}),$$

and

$$\begin{aligned}
&\mathbf{E}_{\theta_0} \exp\left(\mathbf{i}\lambda_j \ln \frac{f\big(\varepsilon_{k+1} + \delta(X_k)\big)}{f\big(\varepsilon_{k+1}\big)}\right)\mathbb{I}_{\{X_k \in \mathbb{D}_j^n\}} = \\
&\mathbf{E}_{\theta_0} \exp\left(\mathbf{i}\lambda_j \ln \frac{f\big(\varepsilon_{k+1} + \delta(\theta_0)\big)}{f\big(\varepsilon_{k+1}\big)}\right)\mathbb{I}_{\{X_k \in \mathbb{D}_j^n\}} + o(n^{-1}) = \\
&\Psi(\lambda_j)\frac{u_j - u_{j-1}}{n}\varphi(\theta_0, \theta_0) + o(n^{-1}).
\end{aligned}$$

Further, by the Markov property, for $k < \ell$

$$\begin{aligned}
&\mathbf{P}_{\theta_0}\big(X_k \in \mathbb{D}_j^n, X_\ell \in \mathbb{D}^n\big) = \mathbf{E}_{\theta_0} \mathbb{I}_{X_k \in \mathbb{D}_j^n} \mathbf{P}_{\theta_0}\big(X_\ell \in \mathbb{D}^n | \mathcal{F}_{\ell-1}\big) = \\
&\mathbf{E}_{\theta_0} \mathbb{I}_{X_k \in \mathbb{D}_j^n} \int_{\mathbb{D}^n} f\Big(x - h(X_{\ell-1})\mathbb{I}_{\{X_{\ell-1} < \theta_0\}} - g(X_{\ell-1})\mathbb{I}_{\{X_{\ell-1} \ge \theta_0\}}\Big)dx \le \\
&C_1 n^{-1} \mathbf{P}_{\theta_0}\big(X_k \in \mathbb{D}_j^n\big) \le C_2 n^{-2},
\end{aligned}$$

where the inequalities hold, since the density $f(x)$ and therefore the invariant density $\varphi(x, \theta_0)$, $x \in \mathbb{R}$ are bounded. Similar bound holds for $k > \ell$ and it follows that

$$\mathbf{P}_{\theta_0}\left(\{X_k \in \mathbb{D}_j^n\} \cap \bigcup_{\ell\neq k, \ell \le n^{1/2}} \{X_\ell \in \mathbb{D}^n\}\right) \le \sum_{\ell \le n^{1/2}, \ell\neq k} \mathbf{P}_{\theta_0}\big(X_k \in \mathbb{D}_j^n, X_\ell \in \mathbb{D}^n\big) \le C_3 n^{-3/2}.$$

Plugging these estimates into (6), we get

$$\mathbf{E}_{\theta_0} e^{\mathbf{i}S_{1,n}} \mathbb{I}_{A_1} = n^{-1/2} \sum_{j=1}^{d} \Psi(\lambda_j)\big(u_j - u_{j-1}\big)\varphi(\theta_0,\theta_0) + o(n^{-1/2}).$$

If we set all λ_j's to zeros, we also obtain

$$\mathbf{P}_{\theta_0}(A_1) = n^{-1/2} \sum_{j=1}^{d} \big(u_j - u_{j-1}\big)\varphi(\theta_0,\theta_0) + o(n^{-1/2}). \tag{7}$$

Further,

$$\mathbf{P}_{\theta_0}(A_0) = 1 - \mathbf{P}_{\theta_0}\left(\bigcup_{\ell \le n^{1/2}} \{X_\ell \in \mathbb{D}^n\}\right) \ge 1 - \sum_{\ell \le n^{1/2}} \mathbf{P}_{\theta_0}\big(X_\ell \in \mathbb{D}^n\big) =$$
$$1 - \sum_{\ell \le n^{1/2}} \sum_{j=1}^{d} \mathbf{P}_{\theta_0}\big(X_\ell \in \mathbb{D}^n_j\big) = 1 - n^{-1/2} \sum_{j=1}^{d} \big(u_j - u_{j-1}\big)\varphi(\theta_0,\theta_0) + o(n^{-1/2}).$$

On the other hand, $\mathbf{P}_{\theta_0}(A_0) \le 1 - \mathbf{P}_{\theta_0}(A_1)$ and in view of (7), it follows that

$$\mathbf{P}_{\theta_0}(A_0) = 1 - n^{-1/2} \sum_{j=1}^{d} \big(u_j - u_{j-1}\big)\varphi(\theta_0,\theta_0) + o(n^{-1/2}). \tag{8}$$

Finally, using (7) and (8), we also have

$$\mathbf{P}_{\theta_0}(A_{2+}) = 1 - \mathbf{P}_{\theta_0}(A_0) - \mathbf{P}_{\theta_0}(A_1) = o(n^{-1/2}).$$

Assembling all parts together, we obtain the asymptotic

$$\mathbf{E}_{\theta_0} e^{\mathbf{i}S_{1,n}} = \mathbf{P}_{\theta_0}(A_0) + \mathbf{E}_{\theta_0} e^{\mathbf{i}S_{1,n}} \mathbb{I}_{A_1} + \mathbf{E}_{\theta_0} e^{\mathbf{i}S_{1,n}} \mathbb{I}_{A_{2+}} =$$
$$1 + n^{-1/2}\left(\sum_{j=1}^{d} \Big(\Psi(\lambda_j) - 1\Big)\big(u_j - u_{j-1}\big)\varphi(\theta_0,\theta_0)\right) + o(n^{-1/2}),$$

and, in turn,

$$\lim_n \left| \Big(\mathbf{E}_{\theta_0} e^{\mathbf{i}S_{1,n}}\Big)^{n^{1/2}} - e^{H(\lambda)} \right| = 0.$$

The claim now follows from (5).

2.2 Equicontinuity

The next step is to show that for some $C>0$

$$\mathbf{E}_{\theta_0}\left(Z_n^{1/2}\left(u_2\right)-Z_n^{1/2}\left(u_1\right)\right)^2\leq C\,\left|u_2-u_1\right|. \tag{9}$$

As in Lemma 2.4 in [4], for e.g. $u_2>u_1>0$, (4) gives

$$\begin{aligned}
&\mathbf{E}_{\theta_0}\left(Z_n^{1/2}\left(u_2\right)-Z_n^{1/2}\left(u_1\right)\right)^2\leq \mathbf{E}_{\theta_0+u_1/n}\ln\frac{Z_n\left(u_1\right)}{Z_n\left(u_2\right)}\\
&\leq \mathbf{E}_{\theta_0+u_1/n}\sum_{j=0}^{n-1}\left|\ln\frac{f\left(\varepsilon_{j+1}\right)}{f\left(\delta\left(X_j\right)+\varepsilon_{j+1}\right)}\right|\,\mathbb{1}_{\left\{\theta_0+\frac{u_1}{n}\leq X_j<\theta_0+\frac{u_2}{n}\right\}}\\
&=\mathbf{E}_{\theta_0+u_1/n}\sum_{j=0}^{n-1}\mathbf{E}_{\theta_0+u_1/n}\left(\left|\ln\frac{f\left(\varepsilon_{j+1}\right)}{f\left(\delta\left(X_j\right)+\varepsilon_{j+1}\right)}\right|\Big|\mathcal{F}_j\right)\,\mathbb{1}_{\left\{\theta_0+\frac{u_1}{n}\leq X_j<\theta_0+\frac{u_2}{n}\right\}}\\
&=\mathbf{E}_{\theta_0+u_1/n}\sum_{j=0}^{n-1}J\left(\delta\left(X_j\right)\right)\,\mathbb{1}_{\left\{\theta_0+\frac{u_1}{n}\leq X_j<\theta_0+\frac{u_2}{n}\right\}}\\
&=n\int_{\theta_0+\frac{u_1}{n}}^{\theta_0+\frac{u_2}{n}}J\left(\delta\left(x\right)\right)\,\varphi\left(x,\theta_0+u_1/n\right)\,\mathrm{d}x\leq C\,\left|u_2-u_1\right|,
\end{aligned}$$

as required.

2.3 Large deviations estimate

Finally we shall prove that for any $p>0$ there exists a constant $C_p>0$ such that

$$\mathbf{E}_{\theta_0}Z_n^{1/2}\left(u\right)\leq\frac{C_p}{|u|^p}. \tag{10}$$

We shall only sketch the proof, as most of the arguments can be directly adopted from the proof of Lemma 2.2 in [3] or Lemma 2.5, [4]. Note that for any $c>0$,

$$\begin{aligned}
&\mathbf{E}_{\theta_0}Z_n^{1/2}(u)=\mathbf{E}_{\theta_0}Z_n^{1/2}(u)\mathbb{1}_{\{Z_n^{1/2}(u)>e^{-c|u|}\}}+\mathbf{E}_{\theta_0}Z_n^{1/2}(u)\mathbb{1}_{\{Z_n^{1/2}(u)\leq e^{-c|u|}\}}\leq\\
&\big(\mathbf{E}_{\theta_0}Z_n(u)\big)^{1/2}\mathbf{P}_{\theta_0}^{1/2}\big(Z_n^{1/2}(u)>e^{-c|u|}\big)+e^{-c|u|}=\mathbf{P}_{\theta_0}^{1/2}\big(\ln Z_n^{1/2}\left(u\right)>-c\left|u\right|\big)+e^{-c|u|}
\end{aligned} \tag{11}$$

and hence it suffices to show that for some $c>0$,

$$\mathbf{P}_{\theta_0}\big(\ln Z_n^{1/2}\left(u\right)>-c\left|u\right|\big)\leq\frac{C_p}{|u|^p},\quad p>0.$$

For $u>0$ (and similarly for $u<0$),

$$\mathbf{P}_{\theta_0}\left\{\ln Z_n^{1/2}\left(u\right)>-c\left|u\right|\right\}$$
$$=\mathbf{P}_{\theta_0}\left\{\sum_{j=0}^{n-1}\ln\left[\frac{f\left(\delta\left(X_j\right)+\varepsilon_{j+1}\right)}{f\left(\varepsilon_{j+1}\right)}\right]\,1\!\!1_{\{\theta_0\leq X_j<\theta_0+u/n\}}>-2cu\right\}. \qquad (12)$$

Let $\ell\left(x,y\right):=\ln\left[\frac{f(\delta(x)+y)}{f(y)}\right]$ and introduce the notations

$$G\left(\delta\right)=-\ln H\left(\delta\right),$$
$$S_n^{(1)}=\sum_{j=0}^{n-1}\ell\left(X_j,\varepsilon_{j+1}\right)\,1\!\!1_{\{X_j\in\mathbb{B}^n\}},\quad S_n^{(2)}=\sum_{j=0}^{n-1}G\left(\delta\left(X_j\right)\right)\,1\!\!1_{\{X_j\in\mathbb{B}^n\}},$$

where $\mathbb{B}^n=[\theta_0,\theta_0+u/n]$ and

$$H\left(\delta\right):=\int_{-\infty}^{\infty}\left(\frac{f\left(\delta+y\right)}{f\left(y\right)}\right)^{1/2}f\left(y\right)\,\mathrm{d}y$$

is the Hellinger integral of order 1/2. By the Jensen inequality for all $\delta\neq 0$, $H(\delta)<1$ and hence $G\left(\delta\right)>0$.

Further, we have the following identity

$$\mathbf{E}_{\theta_0}e^{\frac{1}{2}S_n^{(1)}+S_n^{(2)}}=1. \qquad (13)$$

Indeed

$$\mathbf{E}_{\theta_0}e^{\frac{1}{2}S_n^{(1)}+S_n^{(2)}}=\mathbf{E}_{\theta_0}e^{\frac{1}{2}S_{n-1}^{(1)}+S_{n-1}^{(2)}}\mathbf{E}_{\theta_0}\left(e^{\left[\frac{1}{2}\ell(X_{n-1},\varepsilon_n)+G(\delta(X_{n-1}))\right]1\!\!1_{\{X_{n-1}\in\mathbb{B}^n\}}}\,\middle|\,\mathcal{F}_{n-1}\right)$$

and

$$\begin{aligned}&\mathbf{E}_{\theta_0}\left(e^{\frac{1}{2}\ell(X_{n-1},\varepsilon_n)1\!\!1_{\{X_{n-1}\in\mathbb{B}^n\}}}\,\middle|\,\mathcal{F}_{n-1}\right)=\\ &\quad\mathbf{E}_{\theta_0}\left(e^{\frac{1}{2}\ell(X_{n-1},\varepsilon_n)}\,\middle|\,\mathcal{F}_{n-1}\right)1\!\!1_{\{X_{n-1}\in\mathbb{B}^n\}}+1\!\!1_{\{X_{n-1}\notin\mathbb{B}^n\}}=\\ &\quad\int_{-\infty}^{\infty}\left(\frac{f\left(\delta\left(X_{n-1}\right)+y\right)}{f\left(y\right)}\right)^{1/2}f\left(y\right)\mathrm{d}y\;1\!\!1_{\{X_{n-1}\in\mathbb{B}^n\}}+1\!\!1_{\{X_{n-1}\notin\mathbb{B}^n\}}=\\ &\quad\exp\Big(-G\left(\delta\left(X_{n-1}\right)\right)\;1\!\!1_{\{X_{n-1}\in\mathbb{B}^n\}}\Big).\end{aligned}$$

Hence

$$\mathbf{E}_{\theta_0}\left(e^{\left[\frac{1}{2}\ell(X_{n-1},\varepsilon_n)+G(\delta(X_{n-1}))\right]\mathbb{1}_{\{X_{n-1}\in\mathbb{B}^n\}}}\middle|\mathcal{F}_{n-1}\right)=1$$

and (13) follows. Now we have

$$\mathbf{P}_{\theta_0}\left\{\sum_{j=0}^{n-1}\ell\left(X_j,\varepsilon_{j+1}\right)\ \mathbb{1}_{\{X_j\in\mathbb{B}^n\}}>-2cu\right\}=\mathbf{P}_{\theta_0}\left\{\frac{1}{2}S_n^{(1)}+S_n^{(2)}-S_n^{(2)}>-cu\right\}$$
$$\leq\mathbf{P}_{\theta_0}\left\{\frac{1}{2}S_n^{(1)}+S_n^{(2)}>\frac{1}{2}cu\right\}+\mathbf{P}_{\theta_0}\left\{-S_n^{(2)}>-\frac{3}{2}cu\right\}\leq e^{-\frac{1}{2}cu}+\mathbf{P}_{\theta_0}\left\{S_n^{(2)}<\frac{3}{2}cu\right\}$$

where we used (13). In view of (11) and (12), it is left to show that for all $p>1$,

$$\mathbf{P}_{\theta_0}\left\{\sum_{j=0}^{n-1}G\left(\delta\left(X_j\right)\right)\mathbb{1}_{\{X_j\in\mathbb{B}^n\}}<\frac{3}{2}cu\right\}\leq\frac{C_p}{|u|^p}.$$

Following [3], we shall split the consideration into the cases $u<n^s$ and $n^s\leq u<n\left(\beta-\alpha\right)$, for some $s\in(0,1)$.

To this end, note that the Hellinger integral $H(\delta)$ is a continuous function of δ:

$$\big(H(\delta)-H(\delta+\eta)\big)^2=\left(\int_{-\infty}^{\infty}\left[\left(\frac{f(\delta+y)}{f(y)}\right)^{1/2}-\left(\frac{f(\delta+\eta+y)}{f(y)}\right)^{1/2}\right]f(y)dy\right)^2\leq$$
$$\int_{-\infty}^{\infty}\left(\left(\frac{f(\delta+y)}{f(y)}\right)^{1/2}-\left(\frac{f(\delta+\eta+y)}{f(y)}\right)^{1/2}\right)^2f(y)dy=$$
$$2-2\int_{-\infty}^{\infty}\sqrt{f(\delta+y)f(\delta+\eta+y)}dy=\int_{-\infty}^{\infty}\left(\sqrt{f(y)}-\sqrt{f(\eta+y)}\right)^2dy\leq$$
$$\int_{-\infty}^{\infty}\big|f(y)-f(\eta+y)\big|dy\xrightarrow{\eta\to0}0$$

where we used LeCam's inequality for the Hellinger and the total variation distances and the convergence holds by Scheffe's lemma.

By continuity of $G(\delta)=-\ln H(\delta)$ and since $G(\delta)>0$ for all $\delta\neq0$, the assumption (**a2**) implies that for $u<n^s$

$$G\left(\delta\left(X_j\right)\right)\mathbb{1}_{\{X_j\in\mathbb{B}^n\}}\geq\inf_{\theta_0\leq v\leq\theta_0+n^{s-1}}G\left(\delta\left(v\right)\right)\geq c_0$$

with some constant $c_0>0$ and

$$\mathbf{P}_{\theta_0}\left\{S_n^{(2)}<\frac{3}{2}cu\right\}\leq\mathbf{P}_{\theta_0}\left\{\sum_{j=0}^{n-1}\mathbb{1}_{\{X_j\in\mathbb{B}^n\}}<\frac{3}{2}\frac{c}{c_0}\,u\right\}.$$

Now let $\eta_j(u) = \mathbf{E}_{\theta_0} 1\!\!1_{\{X_j \in \mathbb{B}^n\}} - 1\!\!1_{\{X_j \in \mathbb{B}^n\}}$. Since the density $f(x)$ is continuous and positive, so is the invariant density $\varphi(x, \theta_0)$ and

$$S_n^{(3)} = \sum_{j=0}^{n-1} \mathbf{E}_{\theta_0} 1\!\!1_{\{X_j \in \mathbb{B}^n\}} = n \int_{\theta_0}^{\theta_0 + u/n} \varphi(x, \theta_0)\, \mathrm{d}x \geq C' u,$$

with a positive constant C'. Then

$$\begin{aligned} \mathbf{P}_{\theta_0} \left\{ \sum_{j=0}^{n-1} 1\!\!1_{\{X_j \in \mathbb{B}^n\}} < \frac{3}{2} \frac{c}{c_0} u \right\} &= \mathbf{P}_{\theta_0} \left\{ -\sum_{j=0}^{n-1} \eta_j(u) < -\left(S_n^{(3)} - \frac{3}{2} \frac{c}{c_0} u \right) \right\} \\ &\leq \mathbf{P}_{\theta_0} \left\{ \sum_{j=0}^{n-1} \eta_j(u) > \kappa u \right\} \leq \frac{\mathbf{E}_{\theta_0} \left| \sum_{j=0}^{n-1} \eta_j(u) \right|^{2p}}{|\kappa u|^{2p}}, \end{aligned}$$

where we chose c small enough, so that $C' - \frac{3}{2} c/c_0 = \kappa > 0$. Using the geometric mixing property (**a4**) and an appropriate version of Rosenthal's inequality as in Lemma 2.2 [3], we get

$$\mathbf{E}_{\theta_0} \left| \sum_{j=0}^{n-1} \eta_j(u) \right|^{2p} \leq C(p)\ |u|^p$$

which yields (10) for $|u| < n^s$. The complementary case, $n^s \leq |u| \leq (\beta - \alpha) n$ is treated exactly as in Lemma 2.2, [3] or Lemma 2.5, [4].

3 Discussion

Theorem 1.1 can be directly generalized to the multi-threshold autoregression

$$X_{j+1} = \sum_{k=0}^{K} h_k(X_j)\, 1\!\!1_{\{\theta_k < X_j \leq \theta_{k+1}\}} + \varepsilon_{j+1}, \quad j = 0, 1, \ldots, n,$$

where $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_K)$ is the unknown parameter (and $\theta_0 = -\infty$ and $\theta_{K+1} = \infty$ are set). As in (1), (ε_j) are independent random variables with known density $f(x) > 0, x \in \mathbb{R}$ and the functions $h_k(\cdot)$ and $f(\cdot)$ are continuous and such that (X_j) is geometrically mixing. Assume that $\theta_k \in (\alpha_k, \beta_k)$, where $\beta_k < \alpha_{k+1}$.

For all sufficiently large n and $u_k \geq 0$, the normalized likelihood ratio is given by

$$
\begin{aligned}
Z_n(\mathbf{u}) &= \prod_{j=0}^{n-1} \frac{f\left(X_{j+1} - \sum_{k=0}^{K} h_k(X_j)\, \mathbb{1}_{\left\{\theta_k + \frac{u_k}{n} < X_j \leq \theta_{k+1} + \frac{u_{k+1}}{n}\right\}}\right)}{f\left(X_{j+1} - \sum_{k=0}^{K} h_k(X_j)\, \mathbb{1}_{\{\theta_k < X_j \leq \theta_{k+1}\}}\right)} \\
&= \prod_{j=0}^{n-1} \frac{f\left(\sum_{k=0}^{K} h_k(X_j)\left[\mathbb{1}_{\{\theta_k < X_j \leq \theta_{k+1}\}} - \mathbb{1}_{\left\{\theta_k + \frac{u_k}{n} < X_j \leq \theta_{k+1} + \frac{u_{k+1}}{n}\right\}}\right] + \varepsilon_{j+1}\right)}{f(\varepsilon_{j+1})} \\
&= \prod_{j=0}^{n-1} \frac{f\left(\sum_{k=1}^{K} \left[h_{k-1}(X_j) - h_k(X_j)\right] \mathbb{1}_{\left\{\theta_k < X_j \leq \theta_k + \frac{u_k}{n}\right\}} + \varepsilon_{j+1}\right)}{f(\varepsilon_{j+1})},
\end{aligned}
$$

and thus

$$
\begin{aligned}
\ln Z_n(\mathbf{u}) &= \sum_{j=0}^{n-1} \ln \frac{f\left(\sum_{k=1}^{K} \left[h_{k-1}(X_j) - h_k(X_j)\right] \mathbb{1}_{\left\{\theta_k < X_j \leq \theta_k + \frac{u_k}{n}\right\}} + \varepsilon_{j+1}\right)}{f(\varepsilon_{j+1})} = \\
&\sum_{k=1}^{K} \sum_{j=0}^{n-1} \ln \frac{f\left(\delta_k(X_j) + \varepsilon_{j+1}\right)}{f\left(\varepsilon_{j+1}\right)} \mathbb{1}_{\left\{\theta_k < X_j \leq \theta_k + \frac{u_k}{n}\right\}},
\end{aligned}
$$

where $\delta_k(x) := h_{k-1}(x) - h_k(x)$. Using the same approach as in the proof of Theorem 1.1, it can be seen that

$$
\ln Z_n(\mathbf{u}) \quad \Longrightarrow \quad \sum_{k=1}^{K} \sum_{l=1}^{N_k^+(u_k)} \ln \frac{f\left(\varepsilon_{k,l}^+ + \delta_k(\theta_k)\right)}{f\left(\varepsilon_{k,l}^+\right)},
$$

where $N_k^+(u_k), u_k \geq 0$ are independent Poisson processes with intensities $\varphi(\theta_k, \theta_k)$ and $\varepsilon_{k,l}^+$ are i.i.d. random variables with the density f. Similar asymptotic is obtained for $u_k < 0$. Consequently the limit likelihood ratio is a product on K independent one-dimensional copies of the process defined (2) (with θ_0 replaced by θ_k's) and the corresponding Bayes estimates $\widetilde{\theta}_{k,n},\ k = 1, ..., K$ are asymptotically independent with the asymptotic distribution as in Theorem 1.1.

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